Math 253y – Problem Set 3

Due on Thursday, May 8

Problem 1. (a) *Order statistics of i.i.d. uniforms.* Let $U_1, \dots, U_n$ be independent with Uniform$(0,1)$ distribution and let V_k^n be the kth smallest number in $\{U_1, \dots, U_n\}$. Let T_n be the time of the nth arrival in a rate λ Poisson process. Show that

$$(V_1^n, \dots, V_n^n) \stackrel{\mathcal{L}}{=} (T_1/T_{n+1}, \dots, T_n/T_{n+1}).$$

One can use this result to study the order statistics and their spacings. Set $\lambda = 1$ and let $V_0^n = 0, V_{n+1}^n = 1$.

(b) $nV_k^n \xrightarrow{\mathcal{L}} T_k$ (Smirnov, 1949).

(c) $n^{-1}\#\{k : n(V_k^n - V_{k-1}^n) > x\} \to e^{-x}$ in probability (Weiss, 1955).

(d) $(n/\log n)\max_k(V_k^n - V_{k-1}^n) \to 1$ in probability.

(e) $n^2 \min_k(V_k^n - V_{k-1}^n) \xrightarrow{\mathcal{L}} T_1$.

Problem 2. (a) *Poisson thinning.* Let N have Poisson distribution with mean λ and let $\xi_1, \xi_2, \dots$ be an independent i.i.d. sequence with $\mathbf{P}(\xi_n = i) = p_i$ for $i = 1, \dots, k$. Let

$$N_i \;=\; \#\{n \le N : X_n = i\}, \qquad i = 1, \dots, k.$$

Show that $N_1, \dots, N_k$ are independent and N_i is Poisson with mean λp_i.

(b) A *Poisson process on a measure space* $(S, \mathcal{S}, \mu)$ is a random map $m : \mathcal{S} \to \{0, 1, \dots\}$ that for each ω is a measure, and that has the following property: if $A_1, \dots, A_n$ are disjoint sets with $\mu(A_i) < \infty$, then $m(A_1), \dots, m(A_k)$ are independent Poissons with means $\mu(A_1), \dots, \mu(A_k)$. One calls μ the *mean measure* of the process. Prove such an object exists by giving an explicit construction.

Hint: First consider the case $\mu(S) < \infty$ and make use of Poisson thinning. How might one extend the construction to infinite measure spaces?

One recovers the familiar Poisson process as the case $S = [0, \infty)$, $\mathcal{S} =$ Borel sets, $\mu =$ Lebesgue measure; we described it in terms of the counting function $N_t = m\big([0, t]\big)$, $t \ge 0$.

Problem 3. (a) Let $\mathcal{G} \subset \mathcal{F}$ be σ-subfields and X a r.v. with finite variance. Show that

$$\mathbf{E}(X - \mathbf{E}X|\mathcal{F})^2 + \mathbf{E}(\mathbf{E}X|\mathcal{F} - \mathbf{E}X|\mathcal{G})^2 \;=\; \mathbf{E}(X - \mathbf{E}X|\mathcal{G})^2.$$

Dropping the second term on the left yields an interesting inequality. Geometrically it says the larger the subspace, the closer the projection; statistically it says the more information, the smaller the mean square error.

(b) Let $\mathbf{Var}(X|\mathcal{F}) = \mathbf{E}X^2|\mathcal{F} - (\mathbf{E}X|\mathcal{F})^2$. Show that

$$\mathbf{Var}(X) \;=\; \mathbf{E}\,\mathbf{Var}(X|\mathcal{F}) + \mathbf{Var}(\mathbf{E}X|\mathcal{F}).$$

(c) Let $Y_1, Y_2, \dots$ be i.i.d. with mean μ and variance σ^2, N an independent nonnegative integer-valued r.v. with $\mathbf{E}N^2 < \infty$, and $X = Y_1 + \cdots + Y_N$. Show that

$$\mathbf{Var}(X) \;=\; \sigma^2\mathbf{E}N + \mu^2\,\mathbf{Var}(N).$$

For intuition on this formula, think about the two special cases in which N or Y is constant.

Problem 4. Let $\xi_{n,k}$ be i.i.d. with $\mathbf{P}(\xi_{n,k} = 2) = p$ and $\mathbf{P}(\xi_{n,k} = 0) = 1 - p$ for some $0 < p < 1$. Define the branching process X_n by $X_0 = 1$ and

$$X_{n+1} = \sum_{k=1}^{X_n} \xi_{n,k}.$$

(a) For which values of p is X_n a martingale? sub-martingale? super-martingale?

(b) Show that $M_n = (2p)^{-n} X_n$ is a martingale.

(c) Prove that if $p \le 1/2$ then a.s. $X_n \to 0$, i.e. the population dies out.

(d) Prove that if $p > 1/2$ then with positive probability $\lim M_n > 0$. *Hint:* Use 3(c).

(e) What is the survival probability in the latter case?

Problem 5. *Asymmetric simple random walk.* Fix $0 < p < 1/2$, let $X_1, X_2, \dots$ be i.i.d. with $\mathbf{P}(X_1 = 1) = p$ and $\mathbf{P}(X_1 = -1) = 1 - p$, and consider $S_n = X_1 + \cdots + X_n$.

(a) Find $\gamma > 1$ so that $M_n = \gamma^{S_n}$ is a martingale.

(b) For integers $a < 0 < b$, let $T = \inf\{n : S_n \in \{a, b\}\}$. Show that $\mathbf{E}T < \infty$.

(c) Compute $\mathbf{P}(S_T = b)$.

(d) Determine the distribution of $\sup_n S_n$.

(e) Now let $T = \inf\{n : S_n = a\}$ and compute $\mathbf{E}T$. *Hint:* Use the martingale $S_n + (1-2p)n$. Justify all steps.

Problem 6. Consider the proverbial monkey at a typewriter, typing an i.i.d. sequence of letters uniformly distributed in $\{\mathrm{A}, \dots, \mathrm{Z}\}$. What is the expected amount of time (in keystrokes) until he first consecutively types "ABRACADABRA"?

Hint: Suppose a new gambler arrives on the scene just before each keystroke. He bets \$1 that the next letter will be "A". If he loses, he leaves; if he wins, the house pays him \$26 (which is fair) and he immediately bets it all on the event that the next letter will be "B". If he loses, he leaves; if he wins, he stakes his fortune of $\$26^2$ on the event that the next letter will be "R", and so on through "ABRACADABRA". Now consider the house's net winnings at the relevant time. Fully justify any use of optional stopping!

Problem 7. (a) Recall *Pólya's urn scheme.* An urn initially contains balls of two (or more) colors; a ball is drawn at random and replaced along with an additional ball of the same color; this procedure is then repeated. Prove that the sequence of added balls is exchangeable, i.e. its law is invariant under finite permutations.

(b) Suppose $X_1, X_2, \dots$ are exchangeable real-valued random variables with $\mathbf{E}X_1^2 < \infty$. Prove that $\mathbf{E}X_1 X_2 \ge 0$.

Problem 8. *Brownian bridge.* Let $\big(B(t),\, t \ge 0\big)$ be a Brownian motion on the line.

(a) Prove that the process $\big(X(t),\, 0 \le t \le 1\big)$ defined by $X(t) = B(t) - tB(1)$ is independent of $B(1)$.

(b) Argue informally that the law of $\big(X(t),\, 0 \le t \le 1\big)$ can be interpreted as that of $\big(B(t),\, 0 \le t \le 1\big)$ conditioned on the event $\big\{B(1) = 0\big\}$. There is no need for additional computations.

(c) $\big(X(t),\, 0 \le t \le 1\big)$ is called a *standard Brownian bridge.* Show it is a mean zero Gaussian process and compute its covariance structure $\mathbf{E}X(s)X(t)$, $0 \le s \le t \le 1$. Show that the increments are not independent.

(d) *Fluctuations of the empirical distribution function.* Let $U_1, U_2, \ldots$ be independent Uniform$(0,1)$ and let $N_n(t) = \#\{k \le n : U_k \le t\}$. Recall that almost surely $N_n(t)/n \to t$, in fact uniformly over $0 \le t \le 1$. Prove that

$$\left(\frac{N_n(t) - nt}{\sqrt{n}},\, 0 \le t \le 1\right) \xrightarrow{\mathcal{L}} \big(X(t),\, 0 \le t \le 1\big)$$

in finite dimensional distributions.

(*Donsker's theorem* asserts convergence in law with respect to a much stronger topology.)

Problem 9. Let $\mathcal{T} = \bigcap_{t>0} \sigma\big(B(s), s > t\big)$ be the tail field of Brownian motion; we saw that it is trivial under $\mathbf{P}_x$ for any $x \in \mathbb{R}^d$, i.e. a tail event $A \in \mathcal{T}$ has $\mathbf{P}_x(A) \in \{0,1\}$. Prove that the probability does not depend on the starting point x. *Hint:* Use the Markov property at $t = 1$.

Problem 10. Let $\big(B(t),\, t \ge 0\big)$ be a Brownian motion on the line.

(a) Show that, for $\sigma > 0$, the process $\big(\exp(\sigma B(t) - \sigma^2 t/2),\, t \ge 0\big)$ is a martingale.

(b) Differentiating with respect to σ at 0, discover martingales that are polynomials in $B(t)$ and t of degrees 2, 3 and 4.

(c) Let $a < 0 < b$ and $T = \min\big\{t \ge 0 : B(t) \in \{a, b\}\big\}$. Compute $\mathbf{P}_0(B(T) = b)$, $\mathbf{E}_0 T$ and $\mathbf{E}_0 T^2$.

(d) Let $a, b > 0$. Prove that

$$\mathbf{P}_0\big(B(t) = at + b \text{ for some } t > 0\big) \;=\; e^{-2ab}.$$