Math 253y – Problem Set 2

Due in class on Monday, March 10

Problem 1. *A converse to the SLLN:* Let $X_1, X_2, \ldots$ be i.i.d. random variables with $\mathbf{E}\,|X_1| = \infty$ and let $S_n = X_1 + \cdots + X_n$. Then

$$\limsup \frac{|S_n|}{n} \;=\; \infty \qquad \text{a.s.}$$

Hint: First show the same for $|X_n|/n$, i.e. that $\mathbf{P}(|X_n| \geq Kn \text{ i.o.}) = 1$ for all $K > 0$.

Problem 2. Given a probability distribution $\mu \in \mathcal{P}(\mathbb{R})$, its nth *empirical distribution* is the *random* distribution

$$\mu_n \;=\; \frac{1}{n}\sum_{i=1}^{n} \delta_{X_i}$$

where $X_1, \ldots, X_n$ are i.i.d. with law μ.

(a) For any Borel set $B \in \mathcal{B}(\mathbb{R})$, we have $\mu_n(B) \to \mu(B)$ a.s.

(b) Let F_n, F be the respective distribution functions of μ_n, μ. Show that in fact $F_n \to F$ uniformly:

$$\sup_{x\in\mathbb{R}} |F_n(x) - F(x)| \;\to\; 0 \qquad \text{a.s.}$$

Hint: Let $\varepsilon > 0$ and choose an integer $k > 1/\varepsilon$ and numbers $x_1 \leq \cdots \leq x_{k-1}$ such that $F(x_j-) \leq j/k \leq F(x_j)$ for $j = 1, \ldots, k-1$.

Problem 3. Let $X_1, X_2, \ldots$ be i.i.d. with $\mathbf{P}(X_1 = \pm 1) = 1/2$ and consider the *random signed harmonic series*

$$\sum_{n=1}^{\infty} \frac{X_n}{n}.$$

(a) The series converges almost surely.

(b) Let S be its sum. Finitely many tosses never determine the sign of S with certainty:

$$\mathbf{P}(S > 0,\; X_1 = x_1, \ldots, X_n = x_n) \;>\; 0$$

for every n and $x_1, \ldots, x_n \in \{\pm 1\}$, and likewise with $S < 0$.

(c) Construct a Lebesgue measurable subset $A \subset [0,1]$ such that

$$0 \;<\; m(A \cap I) \;<\; m(I)$$

for every proper interval $I \subset [0,1]$. (Use the above! No messing around with fat Cantor sets etc.)

Problem 4. (a) Convergence in probability implies convergence in law: if $X_n \xrightarrow{p} X$ then $X_n \xrightarrow{\mathcal{L}} X$.

(b) The converse doesn't even make sense in general if X_n are defined on different probability spaces. It holds, however, when X is a constant $c \in \mathbb{R}$, i.e. $X = c$ a.s.

Problem 5. If X_n, Y_n are independent for $1 \le n \le \infty$ and $X_n \xrightarrow{\mathcal{L}} X_\infty$ and $Y_n \xrightarrow{\mathcal{L}} Y_\infty$, then $X_n + Y_n \xrightarrow{\mathcal{L}} X_\infty + Y_\infty$.

Problem 6. Let $\big(X_1^{(n)}, \dots, X_n^{(n)}\big)$ be uniformly distributed on the unit sphere $\{x \in \mathbb{R}^n : |x| = 1\}$. (There is a unique probability measure supported on the sphere that is invariant under orthogonal transformations of $\mathbb{R}^n$.)

(a) How might you generate such a (pseudo-)random vector on a computer? Assume you can generate independent random variables with any of the usual named distributions.

(b) We have $\sqrt{n} X_1^{(n)} \xrightarrow{\mathcal{L}} \chi$, a standard normal random variable.

(c) Jointly for each k, $\sqrt{n}\big(X_1^{(n)}, \dots, X_k^{(n)}\big) \xrightarrow{\mathcal{L}} (\chi_1, \dots, \chi_k)$, k independent standard normals.

"The coordinates of a random point on an infinite-dimensional sphere are independent normals!"

Problem 7. Let $\mu \in \mathcal{P}(\mathbb{R})$ and $\varphi(t) = \int e^{itx}\mu(dx)$, the characteristic function of μ.

(a) If $\int |x|^n \, dx < \infty$ then φ has a continuous derivative of order n given by $\varphi^{(n)}(t) = \int (ix)^n e^{itx}\mu(dx)$.

(b) For the standard normal distribution $\mu(dx) = (2\pi)^{-1/2} e^{-x^2/2} dx$ one has $\varphi(t) = e^{-t^2/2}$.

(c) If X is standard normal then $\mathbf{E}X^{2n} = (2n)!/2^n n! = (2n-1)(2n-3)\cdots 3 \cdot 1 =: (2n-1)!!$.

Problem 8. *Atoms and characteristic functions.* Let $\mu \in \mathcal{P}(\mathbb{R})$ and $\varphi(t) = \int e^{itx}\mu(dx)$.

(a) Imitate the proof of the inversion formula to show that

$$\mu(\{a\}) \;=\; \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} e^{-ita}\varphi(t)\,dt.$$

(b) Show that

$$\sum_x \mu(\{x\})^2 \;=\; \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} |\varphi(t)|^2\,dt.$$

Hint: Let X, Y be independent with law μ and consider $X - Y$.

(c) Conclude that if $\varphi(t) \to 0$ as $|t| \to \infty$ then μ has no atoms.

(d) Let $X_1, X_2, \dots$ be i.i.d. with $\mathbf{P}(X_1 = 0) = \mathbf{P}(X_1 = 1) = 1/2$; then $\sum_n 2X_n/3^n$ has the *Cantor distribution.* Compute its characteristic function φ and consider $\varphi(3^k\pi)$, $k = 0, 1, 2, \dots$ to argue that the converse to (c) is false.

(e) *The Riemann-Lebesgue lemma:* If $\mu(dx) = f(x)dx$ is absolutely continuous then $\lim_{|t|\to\infty}\varphi(t) = 0$.

Problem 9. Let $X_1, X_2, \dots$ be i.i.d. with mean 0 and variance 1 and let $S_n = X_1 + \cdots + X_n$.

(a) Show that $\mathbf{P}(\sup S_n = \infty) > 0$. *Hint:* Consider the event $\{S_n \ge \sqrt{n} \text{ i.o.}\}$.

(b) Conclude that $\sup S_n = \infty$ a.s. by arguing that the event $\{\sup S_n = \infty\}$ is in the tail σ-field of the sequence (X_n).

Problem 10. Let X, Y be *independent* random variables with joint law *invariant under rotations R_θ* of $\mathbb{R}^2$ about the origin: $R_\theta(X, Y) \stackrel{\mathcal{L}}{=} (X, Y)$.

(a) Assuming further that X, Y have finite variance, conclude that they are normally distributed. (For the rotational invariance assumption, invariance under $R_{\pi/4}$ is actually enough!)

(b) Deduce the same conclusion without assuming finite variance.