Math 253y – Problem Set 2

Due in class on Monday, March 10

**Problem 1.** A converse to the SLLN: Let $X_1, X_2, \ldots$ be i.i.d. random variables with $E|X_1| = \infty$ and let $S_n = X_1 + \cdots + X_n$. Then

$$\lim \sup \frac{|S_n|}{n} = \infty \text{ a.s.}$$

*Hint:* First show the same for $|X_n|/n$, i.e. that $P(|X_n| \geq Kn \text{ i.o.}) = 1$ for all $K > 0$.

**Problem 2.** Given a probability distribution $\mu \in P(\mathbb{R})$, its $n$th empirical distribution is the random distribution

$$\mu_n = \frac{1}{n} \sum_{i=1}^{n} \delta_{X_i}$$

where $X_1, \ldots, X_n$ are i.i.d. with law $\mu$.

(a) For any Borel set $B \in \mathcal{B}(\mathbb{R})$, we have $\mu_n(B) \to \mu(B)$ a.s.

(b) Let $F_n, F$ be the respective distribution functions of $\mu_n, \mu$. Show that in fact $F_n \to F$ uniformly:

$$\sup_{x \in \mathbb{R}} |F_n(x) - F(x)| \to 0 \text{ a.s.}$$

*Hint:* Let $\varepsilon > 0$ and choose an integer $k > 1/\varepsilon$ and numbers $x_1 \leq \cdots \leq x_{k-1}$ such that $F(x_j-) \leq j/k \leq F(x_j)$ for $j = 1, \ldots, k-1$.

**Problem 3.** Let $X_1, X_2, \ldots$ be i.i.d. with $P(X_1 = \pm 1) = 1/2$ and consider the random signed harmonic series

$$\sum_{n=1}^{\infty} \frac{X_n}{n}.$$

(a) The series converges almost surely.

(b) Let $S$ be its sum. Finitely many tosses never determine the sign of $S$ with certainty:

$$P(S > 0, \ X_1 = x_1, \ldots, X_n = x_n) > 0$$

for every $n$ and $x_1, \ldots, x_n \in \{\pm 1\}$, and likewise with $S < 0$.

(c) Construct a Lebesgue measurable subset $A \subset [0, 1]$ such that

$$0 < m(A \cap I) < m(I)$$

for every proper interval $I \subset [0, 1]$. (Use the above! No messing around with fat Cantor sets etc.)

**Problem 4.** (a) Convergence in probability implies convergence in law: if $X_n \overset{p}{\to} X$ then $X_n \overset{L}{\to} X$.

(b) The converse doesn’t even make sense in general if $X_n$ are defined on different probability spaces. It holds, however, when $X$ is a constant $c \in \mathbb{R}$, i.e. $X = c$ a.s.
Problem 5. If \( X_n, Y_n \) are independent for \( 1 \leq n \leq \infty \) and \( X_n \xrightarrow{\mathcal{L}} X_\infty \) and \( Y_n \xrightarrow{\mathcal{L}} Y_\infty \), then \( X_n + Y_n \xrightarrow{\mathcal{L}} X_\infty + Y_\infty \).

Problem 6. Let \( (X_1^{(n)}, \ldots, X_n^{(n)}) \) be uniformly distributed on the unit sphere \( \{ x \in \mathbb{R}^n : |x| = 1 \} \). (There is a unique probability measure supported on the sphere that is invariant under orthogonal transformations of \( \mathbb{R}^n \).)

(a) How might you generate such a (pseudo-)random vector on a computer? Assume you can generate independent random variables with any of the usual named distributions.

(b) We have \( \sqrt{n}X_1^{(n)} \xrightarrow{\mathcal{L}} \chi \), a standard normal random variable.

(c) Jointly for each \( k \), \( \sqrt{n}(X_1^{(n)}, \ldots, X_k^{(n)}) \xrightarrow{\mathcal{L}} (\chi_1, \ldots, \chi_k), \) \( k \) independent standard normals.

“The coordinates of a random point on an infinite-dimensional sphere are independent normals!”

Problem 7. Let \( \mu \in \mathcal{P}(\mathbb{R}) \) and \( \varphi(t) = \int e^{itx} \mu(dx) \), the characteristic function of \( \mu \).

(a) If \( \int |x|^n \, dx < \infty \) then \( \varphi \) has a continuous derivative of order \( n \) given by \( \varphi^{(n)}(t) = \int (ix)^n e^{itx} \mu(dx) \).

(b) For the standard normal distribution \( \mu(dx) = (2\pi)^{-1/2}e^{-x^2/2} \, dx \) one has \( \varphi(t) = e^{-t^2/2} \).

(c) If \( X \) is standard normal then \( \mathbb{E}X^{2n} = (2n)!/2^n n! = (2n - 1)(2n - 3) \cdots 3 \cdot 1 =: (2n - 1)!! \).

Problem 8. Atoms and characteristic functions. Let \( \mu \in \mathcal{P}(\mathbb{R}) \) and \( \varphi(t) = \int e^{itx} \mu(dx) \).

(a) Imitate the proof of the inversion formula to show that

\[
\mu(\{a\}) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} e^{-ita} \varphi(t) \, dt.
\]

(b) Show that

\[
\sum_x \mu(\{x\})^2 = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |\varphi(t)|^2 \, dt.
\]

*Hint:* Let \( X, Y \) be independent with law \( \mu \) and consider \( X - Y \).

(c) Conclude that if \( \varphi(t) \to 0 \) as \( |t| \to \infty \) then \( \mu \) has no atoms.

(d) Let \( X_1, X_2, \ldots \) be i.i.d. with \( \mathbb{P}(X_1 = 0) = \mathbb{P}(X_1 = 1) = 1/2 \); then \( \sum_n 2X_n/3^n \) has the Cantor distribution. Compute its characteristic function \( \varphi \) and consider \( \varphi(3^k \pi), k = 0, 1, 2, \ldots \) to argue that the converse to (c) is false.

(e) *The Riemann-Lebesgue lemma:* If \( \mu(dx) = f(x) \, dx \) is absolutely continuous then \( \lim_{|t| \to \infty} \varphi(t) = 0 \).

Problem 9. Let \( X_1, X_2, \ldots \) be i.i.d. with mean 0 and variance 1 and let \( S_n = X_1 + \cdots + X_n \).

(a) Show that \( \mathbb{P}(\sup S_n = \infty) > 0 \). *Hint:* Consider the event \( \{ S_n \geq \sqrt{n} \text{ i.o.} \} \).

(b) Conclude that \( \sup S_n = \infty \) a.s. by arguing that the event \( \{ \sup S_n = \infty \} \) is in the tail \( \sigma \)-field of the sequence \( (X_n) \).

Problem 10. Let \( X, Y \) be independent random variables with joint law invariant under rotations \( R_\theta \) of \( \mathbb{R}^2 \) about the origin: \( R_\theta(X, Y) \overset{\mathcal{L}}{=} (X, Y) \).

(a) Assuming further that \( X, Y \) have finite variance, conclude that they are normally distributed. (For the rotational invariance assumption, invariance under \( R_{\pi/4} \) is actually enough!)

(b) Deduce the same conclusion without assuming finite variance.