Math 253y – Problem Set 1

Due in class on Wednesday, February 19

Problem 1. Consider the following two probability spaces:

$$\Big(\{0,1\}^{\mathbb{N}},\ \sigma\big(\omega_n, n\in\mathbb{N}\big),\ \textstyle\prod_{\mathbb{N}}\big(\frac{1}{2}\delta_0+\frac{1}{2}\delta_1\big)\Big),$$

i.e. coin tossing or product Bernoulli measure, and

$$\big([0,1],\ \mathcal{B},\ m\big),$$

i.e. uniform or Lebesgue measure on the unit interval. Show that binary expansion

$$\begin{aligned}\{0,1\}^{\mathbb{N}} &\to [0,1]\\ \big(\omega_n\big)_{n\in\mathbb{N}} &\mapsto \sum_{n\in\mathbb{N}}\omega_n 2^{-n}\end{aligned}$$

induces an isomorphism between them in the sense that the mapping is bijective up to a null set, measurable in both directions, and measure preserving. *Hint:* The π-system lemma may be useful.

Problem 2. That a collection $\mathcal{A}$ generates a σ-field $\mathcal{F}$ does not in general guarantee that the values of $\mathbf{P}$ on $\mathcal{A}$ determine its values on $\mathcal{F}$. Give an example of a measurable space $(\Omega,\mathcal{F})$, a collection $\mathcal{A}$ with $\mathcal{F}=\sigma(\mathcal{A})$, and probability measures $\mathbf{P},\mathbf{Q}$ such that $\mathbf{P}(A)=\mathbf{Q}(A)$ for $A\in\mathcal{A}$ but $\mathbf{P}\neq\mathbf{Q}$. *Hint:* This can be done on a space with four outcomes!

Problem 3. Let X be a (real-valued) random variable.

(a) If $1\le q\le p\le\infty$ and $X\in L^p$ then $X\in L^q$; in fact, $\|X\|_q\le\|X\|_p$.

(b) $\lim_{p\to\infty}\|X\|_p=\|X\|_\infty$, including that both sides are finite simultaneously.

Problem 4. Suppose $X\ge 0$ and $\mathbf{E}X^2<\infty$. Prove that

$$\mathbf{P}(X>0)\ \ge\ \frac{(\mathbf{E}X)^2}{\mathbf{E}X^2}.$$

Problem 5. *Discrete distributions:* Suppose a random variable X takes values in a countable set S (where the associated σ-field consists simply of all subsets). Then one can express the distribution of X in terms of the *probability mass function* $p_X(x)=\mathbf{P}(X=x)$, since $\mathbf{P}(X\in A)=\sum_{x\in A}p_X(x)$.

(a) What basic fact was used in the last paragraph?

(b) S-valued random variables $X_1,\ldots,X_n$ are independent if and only if their joint probability mass function on S^n factors as the product of their marginal ones:

$$p_{(X_1,\ldots,X_n)}(x_1,\ldots,x_n)=p_{X_1}(x_1)\cdots p_{X_n}(x_n)$$

(c) If integer-valued random variables X, Y are independent, express p_{X+Y} in terms of p_X, p_Y.

Problem 6. *Absolutely continuous distributions:* A random vector $X = (X_1, \ldots, X_n)$ has *absolutely continuous* distribution on $\mathbb{R}^n$ if $\mathbf{P}(X \in A) = 0$ whenever A is Lebesgue-null. In this case there is an integrable function f_X, its *probability density function*, which gives its distribution via the formula $\mathbf{P}(X \in A) = \int_A f_X$ for every Borel set A. In fact

$$f_X(x) \;=\; \lim_{\varepsilon \searrow 0} \frac{1}{(2\varepsilon)^n} \mathbf{P}\big(|X_i - x_i| < \varepsilon,\; i = 1, \ldots, n\big)$$

for Lebesgue-a.e. $x \in \mathbb{R}^n$.

(a) What two theorems were used in the last paragraph?

(b) In this setting each coordinate X_i is absolutely continuous on $\mathbb{R}$. Express f_{X_i} in terms of f_X.

(c) The coordinates are independent if and only if their joint density factors as the product of their marginal densities:

$$f_X(x) \;=\; f_{X_1}(x_1) \cdots f_{X_n}(x_n).$$

(d) If random variables X, Y are absolutely continuous and independent then $X + Y$ is absolutely continuous. Express f_{X+Y} in terms of f_X, f_Y.

(e) Give an example of absolutely continuous random variables X, Y such that $X + Y$ is not absolutely continuous. Describe the joint distribution of (X, Y) on $\mathbb{R}^2$.

Problem 7. Let $A_1, A_2, \ldots$ be independent events. Show that

(a) $\mathbf{1}_{A_n} \to 0$ in probability if and only if $\mathbf{P}(A_n) \to 0$.

(b) $\mathbf{1}_{A_n} \to 0$ almost surely if and only if $\sum \mathbf{P}(A_n) < \infty$.

Problem 8. We saw that almost sure convergence implies convergence in probability.

(a) Give a simple example to show that the converse is false.

(b) In the special case where Ω is countable and $\mathcal{F}$ consists of all its subsets, show that the two modes of convergence are equivalent.

The last two problems are more challenging but they are classic examples.

Problem 9. *Coupon collector's problem:* Let $X_1^{(n)}, X_2^{(n)}, \ldots$ be i.i.d. uniform on $\{1, \ldots, n\}$. Let

$$T_n = \min\big\{t : \#\{X_1^{(n)}, \ldots, X_t^{(n)}\} = n\big\},$$

the "time to collect all n coupons". Prove that

$$\frac{T_n}{n \log n} \;\to\; 1 \qquad \text{in probability.}$$

Hint: Consider the intervals between successive times when you collect new coupons.

Problem 10. *Head runs:* Let $X_1, X_2, \ldots$ be i.i.d. uniform on $\{0, 1\}$ (coin tossing). Let

$$\ell_n \;=\; \max\{m : X_n = X_{n-1} = \cdots = X_{n-m+1} = 1\},$$

the length of the run of heads at time n, and

$$L_n \;=\; \max_{m \le n} \ell_m,$$

the length of the longest run by time n. Prove that

$$\frac{L_n}{\log_2 n} \;\to\; 1 \qquad \text{a.s.}$$